



Meshless Methods

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Enriched EFG Method

- There are two main categories for enhancing EFG approximations:
 1. Extrinsic enrichment (enriched trial function) where the approximation is enriched by adding functions externally to the EFG approximation:
 1. Extrinsic MLS enrichment
 2. Extrinsic PU enrichment
 2. Intrinsic enrichment (enriched basis) where the enrichment functions are included in the EFG (MLS) basis:
 1. Full enrichment
 2. Radial enrichment

Enriched EFG Method

1.1- Extrinsic MLS enrichment

- In this method, the EFG are augmented to include the leading terms of the near-tip asymptotic expression for the displacement field at a crack tip.
- The approximation takes the form:

$$u_{\alpha}^h(x) = p^T(x) a_{\alpha}(x) + \sum_{J=1}^{n_c} \left[k_1^J Q_{1\alpha}^J(x) + k_2^J Q_{2\alpha}^J(x) \right], \alpha = 1, 2$$

$u_{\alpha}^h(x)$ = the approximation of the exact solution $u_{\alpha}(x)$

$p(x)$ = a complete basis of monomials

$a_{\alpha}(x)$ = the coefficients of the polynomials

k_1^J, k_2^J = the unknown parameters associated with crack J

n_c = the number of cracks in the model

Enriched EFG Method

- The functions Q describe the near-tip displacement field (Anderson-1991):

$$Q_{11}(x) = \frac{1}{2\mu} \sqrt{\frac{r}{2\pi}} \cos\left(\frac{\theta}{2}\right) \left[K - 1 + 2 \sin^2\left(\frac{\theta}{2}\right) \right]$$

$$Q_{12}(x) = \frac{1}{2\mu} \sqrt{\frac{r}{2\pi}} \sin\left(\frac{\theta}{2}\right) \left[K + 1 - 2 \cos^2\left(\frac{\theta}{2}\right) \right]$$

$$Q_{13}(x) = \frac{1}{2\mu} \sqrt{\frac{r}{2\pi}} \sin\left(\frac{\theta}{2}\right) \left[K + 1 + 2 \cos^2\left(\frac{\theta}{2}\right) \right]$$

$$Q_{14}(x) = \frac{-1}{2\mu} \sqrt{\frac{r}{2\pi}} \cos\left(\frac{\theta}{2}\right) \left[K - 1 - 2 \sin^2\left(\frac{\theta}{2}\right) \right]$$

Enriched EFG Method

r = the distance from the crack tip

θ = the angle from the tangent to the crack path at the crack tip

μ = the shear modulus

K = the Kolosov modulus

$$K = \begin{cases} 3-4\nu & \text{plane strain} \\ \frac{3-\nu}{1+\nu} & \text{plane stress} \end{cases}$$

Enriched EFG Method

- The coefficients $a_\alpha(x)$ are functions of the spatial coordinates and are determined by minimizing a weighted discrete L_2 norm:

$$\Pi = \sum_{I=1}^n \frac{1}{2} W(x - x_I) \left[p^T(x) a_\alpha(x) + \sum_{J=1}^{n_c} \left[k_1^J Q_{1\alpha}^J(x) + k_2^J Q_{2\alpha}^J(x) \right] - u_{I\alpha} \right]^2$$

n = the number of points in the support domain of x

$u_{I\alpha}$ = the α component of the nodal value at $x=x_I$

- The stationary condition of Π with respect to $a_\alpha(x)$ leads to:

$$A(x) a_\alpha(x) = \sum_{I=1}^n P_I(x) \left\{ u_{I\alpha}(x) - \sum_{J=1}^{n_c} \left[k_1^J Q_{1\alpha}^J(x) + k_2^J Q_{2\alpha}^J(x) \right] \right\}$$

Enriched EFG Method

$$A(x) = \sum_{I=1}^n W(x-x_I) p(x_I) p^T(x_I)$$

$$P_I(x) = [W(x-x_1)p(x_1) \quad W(x-x_2)p(x_2) \quad \cdots \quad W(x-x_n)p(x_n)]$$

- Solving for $a_\alpha(x)$:

$$a_\alpha(x) = \sum_{I=1}^n A^{-1}(x) P_I(x) \left\{ u_{I\alpha}(x) - \sum_{J=1}^{n_c} [k_1^J Q_{1\alpha}^J(x) + k_2^J Q_{2\alpha}^J(x)] \right\}$$

- The final approximation:

$$u_\alpha^h(x) = \sum_{I=1}^n \phi_I(x) \hat{u}_{I\alpha}(x) + \sum_{J=1}^{n_c} [k_1^J Q_{1\alpha}^J(x) + k_2^J Q_{2\alpha}^J(x)]$$

$$\phi_I(x) = p^T(x) A^{-1}(x) P_I(x)$$

$$\hat{u}_{I\alpha}(x) = u_{I\alpha}(x) - \sum_{J=1}^{n_c} [k_1^J Q_{1\alpha}^J(x) + k_2^J Q_{2\alpha}^J(x)]$$

Enriched EFG Method

1.2- Extrinsic PU enrichment

- The extrinsic basis is smoothly added to the approximation through the partition of unity.
- The enriched approximation based on PU takes the form:

$$u^h(x) = \sum_{I=1}^n \phi_I^k(x) u_I + \sum_{I=1}^n \sum_{J=1}^{m_e} \phi_I^0(x) b_{IJ} q_J(x)$$

u_I = nodal coefficients

b_{IJ} = nodal coefficients

n = number of neighbors of point x

k = the order of the polynomial of the basis used in forming the PU ($k=0$ for shepard PU)

$q(x)$ = the extrinsic basis (Q or simply a function of $\sqrt{r}$)

$\phi_I(x)$ = the partition of unity shape functions

Enriched EFG Method

2.1- Full intrinsic basis enrichment

- The entire near-tip asymptotic displacement field is included in the basis:

$$p^T(x) = \left[1, x, y, \sqrt{r} \cos\left(\frac{\theta}{2}\right), \sqrt{r} \sin\left(\frac{\theta}{2}\right), \sqrt{r} \sin\left(\frac{\theta}{2}\right) \sin(\theta), \sqrt{r} \cos\left(\frac{\theta}{2}\right) \sin(\theta) \right]$$

- The displacement field can be written as:

$$u_{\alpha}^h(x) = \sum_{I=1}^n p^T(x) A^{-1}(x) P_I(x) u_{I\alpha}$$

Enriched EFG Method

1. The method involves no additional unknowns.
2. Because of the increased size of the basis, additional computational effort is required to invert the moment matrix $A(x)$.
3. The domain of influence must be enlarged to achieve regularity of $A(x)$.
4. Using an enriched basis can lead to an ill-conditioned $A(x)$.
5. Two treatments are available:
 1. Gram-Schmidt orthogonalization for diagonalizing A .
 2. Reducing the number of shape function computations.

Enriched EFG Method

2.2- Radial intrinsic basis enrichment

- The basis is defined based on the radial distance from the crack tip, r :

$$p^T(x) = [1, x, y, \sqrt{r}]$$

- In a radial enrichment:
 - The angular variation around the crack tip is smooth.
 - The radial variation of stress is singular.
- Inverting the moment matrix $A(x)$ is much cheaper than for full enrichment.
- No smoothing technique is necessary, because there are no displacement in the radial direction.

Enriched EFG Method

- Enriching the approximation for the entire domain of a problem is generally unnecessary and increases computational expenses. For example, the crack tip singular field is local; it extends almost 10% of the length of the crack from the crack tip.
- To couple linear and enriched approximations, a linear combination over a transition region is used.

Enriched EFG Method

- Enriched EFG approximation:

$$u^h(x) = Ru^{enr}(x) + (1-R)u^{lin}(x)$$

$u^{enr}(x)$ = the enriched approximation

$u^{lin}(x)$ = the linear basis approximation

$$R = \begin{cases} 1 & \text{on the enriched boundary of the coupling region } r=r_1 \\ 0 & \text{on the linear boundary of the coupling region } r=r_2 \end{cases}$$

- Suggested polynomial ramp functions:

- Linear ramp $R = 1 - \xi$

- Quintic ramp $R = 1 - 10\xi^3 + 15\xi^4 - 6\xi^5$

$$\xi = \frac{r - r_1}{r_2 - r_1}$$

Enriched EFG Method

- The coupled approximation is written:

$$u^h(x) = \sum_{I=1}^n \hat{\phi}_I(x) u_I$$

$$\hat{\phi}_I(x) = R\phi_I^{enr}(x) + (1-R)\phi_I^{lin}(x)$$

Enriched EFG Method

1. This method ensures a compatible displacement field.
2. The continuity in the strain field depends on the continuity of the ramp function R :
 - Linear ramp gives discontinuous strains at $r=r_1, r_2$.
 - Quintic ramp provides continuous strain field.
3. If the area of enrichment about a crack tip is circular,
 - The inner radius of the ramp r_1 can begin at the crack tip.
 - The outer radius r_2 needs to be outside the singularity dominated zone to obtain good accuracy.
4. If the enrichment is confined to the crack tip region by simply changing from an enriched to an un-enriched basis, the approximations will not be continuous.

Enriched EFG Method

- While this is not strictly permissible in a Galerkin method, the errors are small if the transition occurs outside the singular stress dominated region ($>0.1a$) so that jump is very small.
- Mixing u^{enr} and u^{lin} was found to work better for the intrinsically enriched basis than for the extrinsic MLS enrichment which is sensitive to discontinuities.
- Extrinsic PU enrichment is added only to those nodes which are near the crack tip. It is necessary to enrich all nodes whose domains of influence include the crack tip. Otherwise, the enrichment is incomplete and the solution is poor.

Enriched EFG Method

- The strong form of linear elastostatics:

$$\begin{cases} \vec{\nabla} \cdot \sigma + b = 0 & \text{in } \Omega \\ u = \bar{u} & \text{on } \Gamma_u \\ \sigma \cdot n = \bar{t} & \text{on } \Gamma_t \end{cases}$$

- The modified variational principle can be written as (according to Liu/Belytschko/Gu -1994):

$$\int_{\Omega} \delta \varepsilon : \sigma d\Omega - \int_{\Omega} \delta u \cdot b d\Omega - \int_{\Gamma_t} \delta u \cdot \bar{t} d\Gamma - \int_{\Gamma_u} \delta t (u - \bar{u}) d\Gamma - \int_{\Gamma_u} \delta u \cdot t d\Gamma = 0$$

- Other ways of imposing the essential boundary condition:
 - Lagrange multipliers
 - Penalty method

Enriched EFG Method

- Using the enriched approximation:

$$u_{\alpha}^h(x) = \sum_{I=1}^n \phi_I(x) \hat{u}_{I\alpha} + \sum_{J=1}^{n_c} \left[k_1^J Q_{1\alpha}^J(x) + k_2^J Q_{2\alpha}^J(x) \right]$$

$$u_{\alpha}^h(x) = \sum_{I=1}^n \phi_I(x) u_{I\alpha}$$

- The discretized system of equations is obtained:

$$Kd = F^{ext}$$

$$K_{m \times m} d_{m \times 1} = f_{m \times 1}^{ext}$$

$$K = \bigcap K_{IJ} \quad F^{ext} = \bigcap F_I^{ext}$$

$$K_{IJ} = \int_{\Omega} B_I^T D B_J d\Omega - \int_{\Gamma_u} \phi_I S N C B_J d\Gamma - \int_{\Gamma_u} B_I^T D^T N^T S \phi_J d\Gamma$$

$$F_I^{ext} = \int_{\Omega} \phi_I b d\Omega + \int_{\Gamma_t} \phi_I \bar{t} d\Gamma - \int_{\Gamma_u} B_I^T D^T N^T S \bar{u} d\Gamma$$

Enriched EFG Method

$$D = \frac{E}{(1+\nu)(1-\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \quad \text{plane strain}$$

$$B_I^T = \begin{bmatrix} \phi_{I,x} & 0 & \phi_{I,y} \\ 0 & \phi_{I,y} & \phi_{I,x} \end{bmatrix}$$

$$N = \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix}$$

$$S = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \quad S_a = \begin{cases} 1 & : \text{if } \bar{u}_a \in \bar{u} \\ 0 & : \text{if } \bar{u}_a \notin \bar{u} \end{cases}, a = x, y$$

Enriched EFG Method

- The augmented trial functions of extrinsic method lead to extra unknowns:

$$\bar{K}_{(m+2n_c) \times (m+2n_c)} \bar{d} = \bar{f}^{ext}$$

$$\bar{K} = \begin{bmatrix} K & K_{I1} & K_{I2} \\ K_{I1}^T & K_{11} & K_{12} \\ K_{I2}^T & K_{12}^T & K_{22} \end{bmatrix}$$

$$\bar{d}^T = [\hat{u} \quad k_1^1 \quad k_2^1 \quad \cdots \quad k_1^J \quad k_2^J]$$

$$\bar{f}^{ext^T} = [f^{ext} \quad f_1^1 \quad f_2^1 \quad \cdots \quad f_1^J \quad f_2^J]$$

Enriched EFG Method

- The interaction term in the stiffness matrix between node I and the mode 1 stress intensity factor K_{I1} :

$$K_{I1} = \int_{\Omega} B_I^T D \hat{B}_1 d\Omega - \int_{\Gamma_u} \phi_I S N C \hat{B}_1 d\Gamma - \int_{\Gamma_u} B_I^T D^T N^T S \hat{\phi}_1 d\Gamma$$

$$\hat{\phi}_1 = [Q_{11}(x) \quad Q_{12}(x)]$$

$$\hat{B}_1 = \begin{bmatrix} Q_{11,x}(x) \\ Q_{12,y}(x) \\ Q_{11,y}(x) + Q_{12,x}(x) \end{bmatrix}$$

Enriched EFG Method

- The term in the stiffness matrix emanating from mode 1 enrichment, K_{11} (and similar forms for all other terms):

$$K_{11} = \int_{\Omega} \hat{B}_1^T D \hat{B}_1 d\Omega - \int_{\Gamma_u} \hat{\phi}_1 S N C \hat{B}_1 d\Gamma - \int_{\Gamma_u} \hat{B}_1^T D^T N^T S \hat{\phi}_1 d\Gamma$$

- The additional term in the force vector due to mode 1 enrichment is:

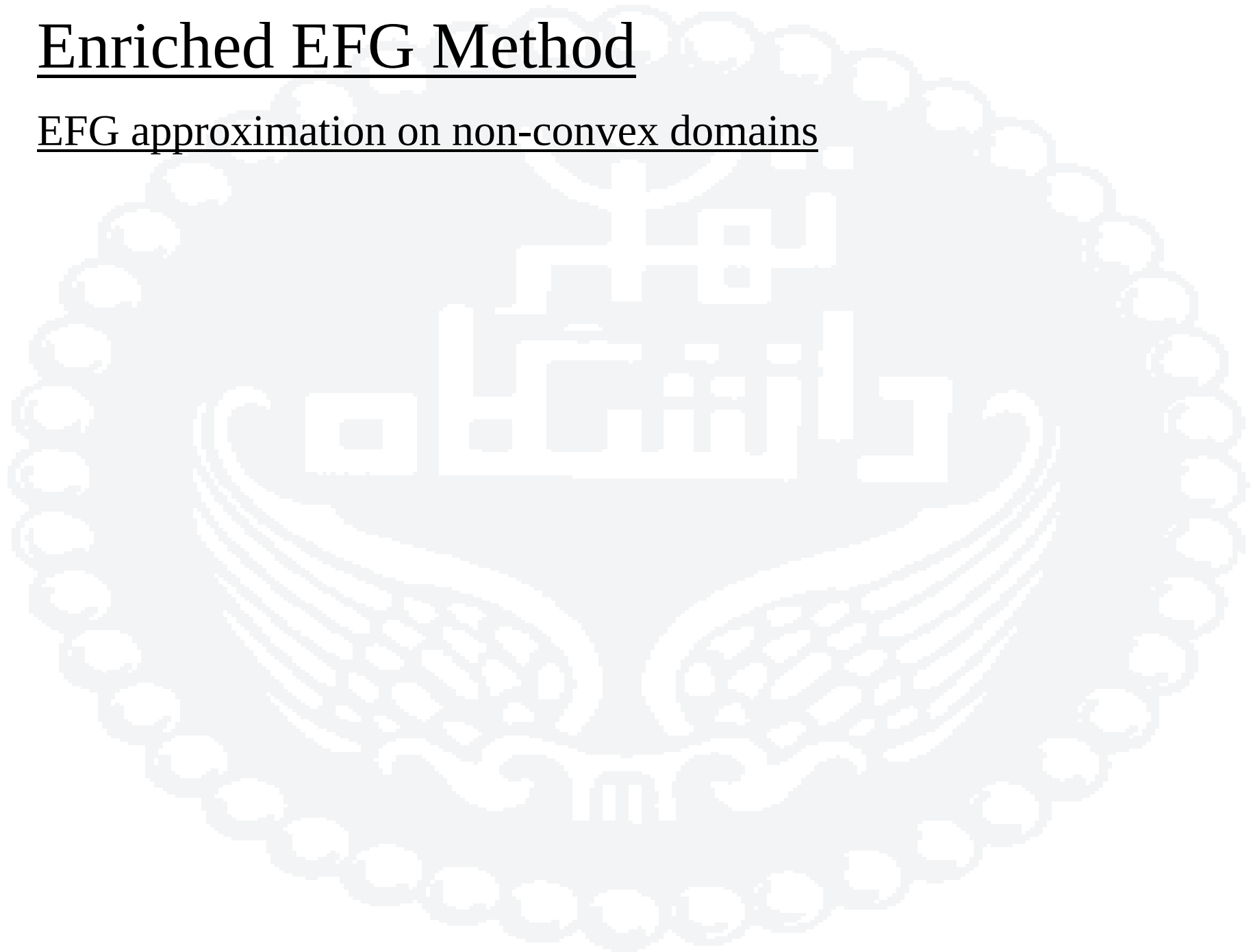
$$f_1^{ext} = \int_{\Omega} \hat{\phi}_1^T b d\Omega + \int_{\Gamma_t} \hat{\phi}_1^T \bar{t} d\Gamma - \int_{\Gamma_u} \hat{B}_1^T D^T N^T S \bar{u} d\Gamma$$

Enriched EFG Method



Enriched EFG Method

EFG approximation on non-convex domains



Enriched EFG Method

EFG approximation on non-convex domains

- Without smoothing, EFG approximations near non-convex boundaries, such as crack tips, may become discontinuous.
- Available smoothing methods are:
 1. The visibility criterion (VC)
 2. The diffraction method
 3. The transparency method
 4. The see-through method (continuous line criterion)

Enriched EFG Method

1. The visibility criterion (VC)

- Proposed by Belytschko – 1994.
- It defines the domain of influence of a node as the field of vision of the node. All boundaries, internal and external, are considered to be opaque so that the field of vision is interrupted when a boundary is encountered.

Enriched EFG Method

1. The visibility criterion (VC)

- A difficulty with the visibility criterion arises for nodes near the tip of a crack, such as node I . A discontinuity in the weight function as well as the shape function along the line AB is observed.

Enriched EFG Method

1. The visibility criterion (VC)

- The visibility criterion leads to discontinuities in shape functions for nodes near non-convex boundaries such as holes and cracks.
- The presence of discontinuities within the domain is undesirable in a Galerkin method, and must be handled with care.

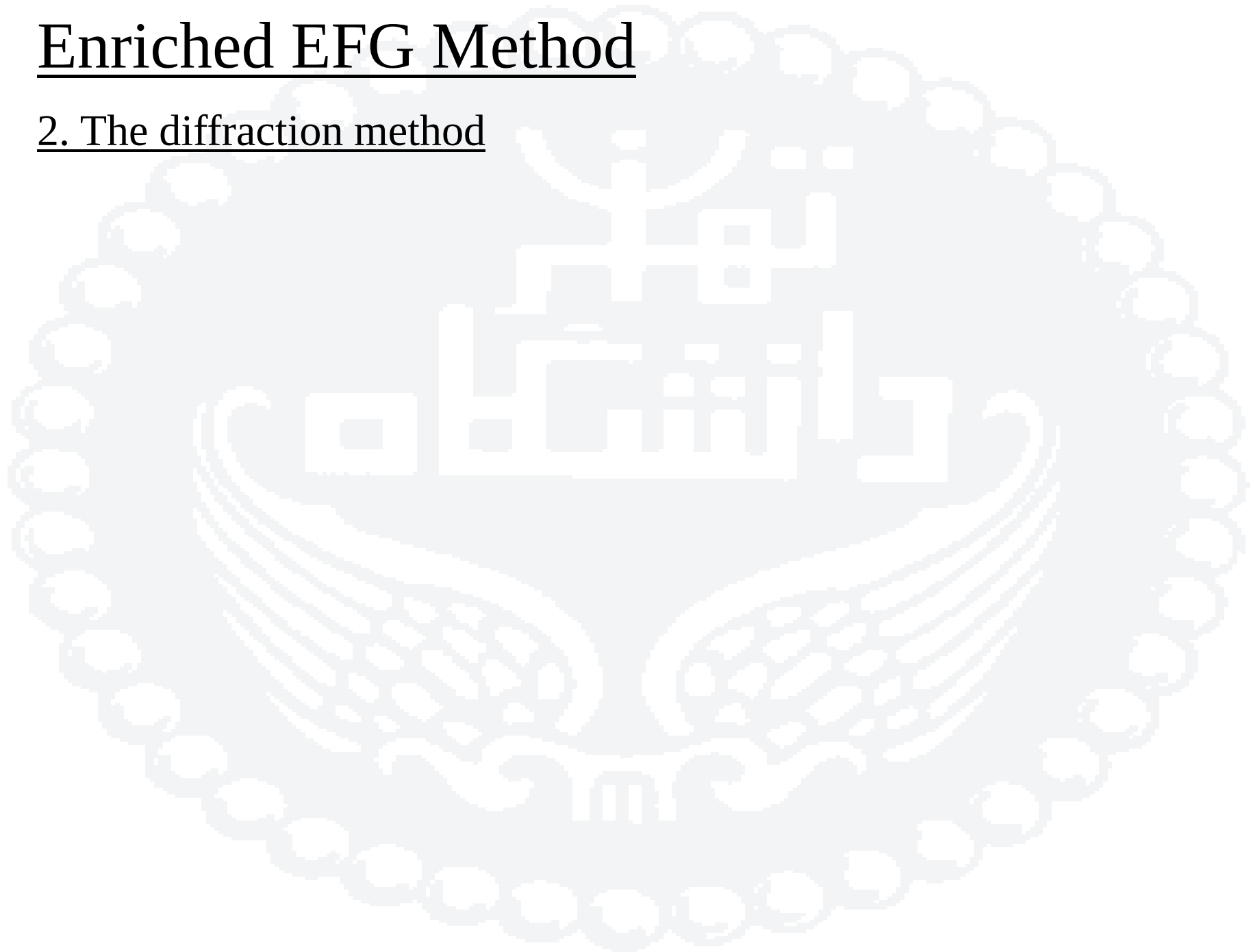
Enriched EFG Method

2. The diffraction method

- Continuous and smooth approximations can be constructed near non-convex boundaries by the diffraction method.
- In this method, the nodal support is wrapped around non-convex boundaries similar to the way light diffracts around sharp corners (also called wrap-around method).
- Consider a case where the ray between the node x_I and a sampling point x intersects a crack, and the tip is within the domain of influence of the node.

Enriched EFG Method

2. The diffraction method



Enriched EFG Method

2. The diffraction method

- The weight function distance d_I is modified by:

$$d_I = \left(\frac{S_1 + S_2(x)}{S_0(x)} \right)^\lambda S_0(x)$$

$$S_1 = \|x_I - x_C\|$$

$$S_2(x) = \|x - x_C\|$$

$$S_0(x) = \|x - x_I\|$$

x_C = crack tip

λ = a parameter to adjust the distance of the support on the opposite side of the crack ($\lambda = 1, 2$ performs well)

Enriched EFG Method

2. The diffraction method

- The method is quite general, and can be used for cracks or interior holes.
- This method also works well for general non-convex boundaries. The tangent point between the node and the non-convex boundary is then used as the wrap-around point x_C .

Enriched EFG Method

3. The transparency method

- In this method, the transparency of the crack varies so that it is completely opaque a short distance from the tip.
- In this way, the field of vision for a node near the crack tip is not abruptly truncated when it reaches the crack tip, but rather diminishes smoothly to zero a short distance from the crack tip.

Enriched EFG Method

3. The transparency method

- The distance parameter d_I in the weight function is modified:

$$d_I(x) = S_0(x) + d_{mI} \left(\frac{S_C(x)}{\bar{S}_C} \right)^\lambda, \lambda \geq 2$$

$$S_0(x) = \|x - x_I\|$$

d_{mI} = the radius of support for node I

$S_C(x)$ = the intersection distance behind the crack tip

$\bar{S}_C = \kappa h$ = the distance behind the crack tip at which complete opacity occurs

h = nodal spacing

κ = a constant $0 < \kappa < 1$

Enriched EFG Method

3. The transparency method

- The shape function is continuous at the crack tip.
- The method does not work well when nodes are placed too close to the crack surface. As a result, it is proposed all nodes should be placed so that the normal distance from the node to the crack surface is greater than roughly $(h/4)$.

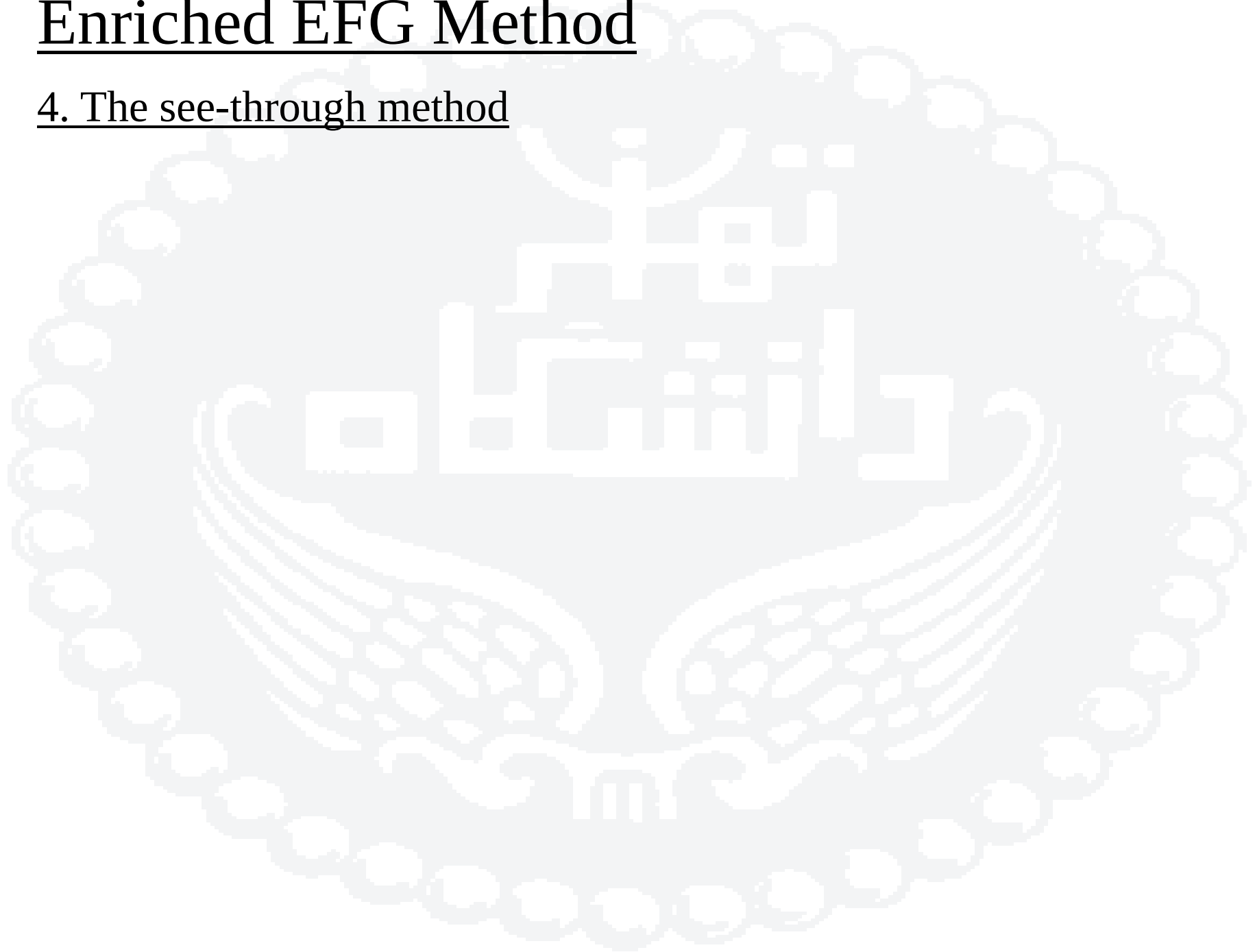
Enriched EFG Method

4. The see-through method

- Proposed by Terry (1994) in his M.Sc. thesis.
- In this method, all or part of the boundary is made completely transparent such that discontinuities are eliminated.
- Duarte/Oden (1996) and Krysl/Belytschko () suggested similar procedures, called the continuous line criterion or contained path criterion (CPC):
 - If a continuous line connecting the node to a point lies entirely within the domain of influence of the node, the point is visible.
 - The method is easy to implement and provides smooth approximations.
 - In modelling cracks, it effectively shortens the crack and leads to inaccurate solutions. However, it does work for cracks when used with the enriched EFG.

Enriched EFG Method

4. The see-through method



Enriched EFG Method

4. The see-through method

- When $n_A \cdot n_B \leq \beta$ then the domain of influence is cut.
 n_A, n_B are the surface normals to the wedge
 β is a cut off value
- A minimum value of $\beta = 0$ ($\omega = 90^\circ$) is recommended.

